# Computer Science 294 Lecture 4 Notes

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# 1 Properties of Effect and Influence

## 1.1 Recap: voting rules, effect, and influence

Last time, we discussed thinking of a boolean function $f : \{\pm 1\}^n \to \{\pm 1\}$ as a voting rule. We had a number of examples, including the Majority, And, Or, Dictator, and Tribes functions. We also defined the influence

$$\mathrm{Inf}_i(f) = \mathbb{P}_{X \sim \{\pm 1\}^n}(f(X) \neq f(X^{\oplus i})), \qquad x^{\oplus i} = x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n)$$

and the effect

$$\mathrm{Eff}_i(f) = \mathbb{P}_{X \sim \{\pm 1\}^n}(f(X) = 1 \mid X_i = 1) - \mathbb{P}_{X \sim \{\pm 1\}^n}(f(X) = 1 \mid X_i = -1) = \widehat{f}(\{i\}).$$

To derive a Fourier representation for the influence, we introduced the derivative

$$D_i f(x) = \frac{f(x^{(i \mapsto 1)}) - f(x^{(i \mapsto -1)})}{2}.$$

If $f$ is a boolean function, then we saw that

$$D_i f(x) = \begin{cases} \pm 1 & \text{if } i \text{ is pivotal} \\ 0 & \text{otherwise.} \end{cases}$$

## 1.2 Fourier representations of effect and influence

From the definition,

$$\mathrm{Inf}_i(f) = \mathbb{E}[(D_i f(X))^2].$$

Recall that the fundamental theorem of boolean functions gives

$$f(x) = \sum_{S \subseteq [n]} \widehat{f}(S) \chi_S(x), \qquad \chi_S = \prod_{i \in S} x_i.$$

**Proposition 1.1.** $D_i f : \{\pm 1\}^n \to \mathbb{R}$ *can be expressed as*

$$D_i f(x) \sum_{S \subseteq [n] : S \ni i} \widehat{f}(S) \prod_{j \in S \setminus \{i\}} x_j.$$

**Example 1.1.** If $f(x) = \frac{1}{2}x - 1x_2 + \frac{1}{2}x_3x_4$, then $D_1 f(x) = \frac{1}{2}x_2$.

*Proof.* $D_i$ is a linear operator:

$$D_i(f+g) = D_i f + D_i g, \qquad D_i(\alpha f) = \alpha D_i f.$$

We can write $f$ as a linear combination of characters, so it suffices to see what $D_i$ does to a single character. In particular, we want to show

$$D_i \chi_S = \begin{cases} \chi_{S \setminus \{i\}} & i \in S \\ 0 & i \notin S. \end{cases}$$

When $i \in S$,

$$\begin{aligned} D_i \prod_{j \in S} x_i &= \frac{\prod_{j \in S \setminus \{i\}} x_j \cdot 1 - \prod_{j \in S \setminus \{i\}} x_j \cdot (-1)}{2} \\ &= \prod_{j \in S \setminus \{i\}} x_j. \end{aligned}$$ □

The definition $\mathrm{Inf}_i(f) = \mathbb{E}[(D_i f(X))^2]$ makes sense for any $f : \{\pm 1\}^n \to \mathbb{R}$, not just boolean functions.

**Lemma 1.1.**

$$\mathrm{Inf}_i(f) = \sum_{S \ni i} \widehat{f}(S)^2.$$

*Proof.*

$$\mathrm{Inf}_i(f) = \langle D_i f, D_i f \rangle$$

Using Parseval's identity,

$$= \sum_{S \subseteq [n]} \widehat{D_i f}(S)^2$$

The previous proposition tells us that if $i \notin S$, $\widehat{D_i f}(S) = \widehat{f}(S \cup \{i\})$. If $S$ contains $i$, then $\widehat{D_i f}(S) = 0$.

$$= \sum_{T \ni i} \widehat{f}(T)^2.$$ □

**Lemma 1.2.**

$$\mathrm{Eff}_i(f) = \mathbb{E}[D_i f(X)].$$

*Proof.*

$$\begin{aligned}
\mathbb{E}_{X\sim\{\pm 1\}^n}[D_i f(X)] &= \langle D_i f(X), 1\rangle \\
&= \widehat{D_i f}(\varnothing) \\
&= \widehat{f}(\{i\}) \\
&= \mathrm{Eff}_i(f).
\end{aligned}$$

□

**Corollary 1.1.** *For any $f : \{\pm 1\}^n \to \{\pm 1\}$ and any coordinate $i$,*

$$\mathrm{Eff}_i(f) \le \mathrm{Inf}_i(f).$$

*Proof.*

$$\mathrm{Eff}_i(f) = \mathbb{E}[D_i f(X)]$$

Since $D_i f$ equals $-1$, $0$, or $+1$,

$$\begin{aligned}
&= \mathbb{E}[(D_i f(X))^2] \\
&= \mathrm{Inf}_i(f).
\end{aligned}$$

□

**Remark 1.1.** We can think of the effect as how likely person $i$ can change the result if they move first, whereas the influence is how likely person $i$ can change the result if they move last. So in general, person $i$ has more power if they move last.

**Remark 1.2.** We have equality if and only if $f$ is monotone in the $i$-th direction.

## 1.3 Total influences and total effects

**Definition 1.1.** The **total influence** of a function $f : \{\pm 1\}^n \to \mathbb{R}$ is

$$\mathbb{I}(f) = \sum_{i\in[n]} \mathrm{Inf}_i(f).$$

**Definition 1.2.** The **total effect** of a function $f : \{\pm 1\}^n \to \mathbb{R}$ is

$$\mathrm{Eff}(f) = \sum_{i\in[n]} \mathrm{Eff}_i(f).$$

**Example 1.2.** Since the Majority function is symmetric,

$$\mathbb{I}(\mathrm{MAJ}_n) = n\,\mathrm{Inf}_1(\mathrm{MAJ}_n) \approx n \cdot \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{n}} = \Theta(\sqrt{n}).$$

**Example 1.3.** The Parity function is symmetric, so

$$\mathbb{I}(\text{Parity}_n) = n \operatorname{Inf}_1(\text{Parity}) = n.$$

**Example 1.4.**

$$\mathbb{I}(\chi_i) = \sum_{j \in [n]} \operatorname{Inf}_j(\chi_i) = \operatorname{Inf}_i(\chi_i) = 1.$$

**Example 1.5.** For the Tribes function, we are taking the Or of $s$ Ands of $w$ voters each. To find the influence of voter 1, observe that voter 1's vote is pivotal only when all other tribes vote False, but all other members of $x_1$'s tribe vote True.

$$\begin{aligned}\mathbb{P}_{X_2,\dots,X_n}(X_1 \text{ is pivotal for Tribes}) &= \mathbb{P}(\text{all other members of 1's Tribes vote True}) \\ &\quad \cdot \mathbb{P}(\text{all other Tribes vote False}) \\ &= \left(\frac{1}{2}\right)^{w-1} \cdot \left(1 - \frac{1}{2^w}\right)^{s-1}\end{aligned}$$

Taking $s \approx \ln 2 \cdot 2^w$,

$$\begin{aligned}&\approx \frac{1}{2^w} \\ &= \Theta\left(\frac{\log n}{n}\right).\end{aligned}$$

So the total influence of Tribes is

$$\mathbb{I}(\text{Tribes}_n) = \Theta(\log n).$$

You may wonder if a voting rule which is relatively stable can give all voters low influence.

**Theorem 1.1** (KKL). *If $f : \{\pm 1\}^n \to \{\pm 1\}$ and $\operatorname{Var}(f) = \Omega(1)$, then there exists a coordinate $i$ with influence $\operatorname{Inf}_i \geq \Omega(\frac{\log n}{n})$.*

To prove this, we will need some more machinery.

## 1.4 Sensitivity and interpretations of total influence

**Definition 1.3.** The **sensitivity** of $f$ at $x$ is

$$\operatorname{Sens}_f(x) = \#\text{ of pivotal coordinates on } x.$$

**Lemma 1.3.**

$$\mathbb{I}(f) = \mathbb{E}_{X \sim \{\pm 1\}^n}[\operatorname{Sens}_f(X)].$$

So the total influence equals the average sensitivity.

*Proof.*

$$
\begin{aligned}
\mathbb{I}(f) &= \sum_{i\in[n]} \mathrm{Inf}_i(f) \\
&= \sum_{i=1}^{n} \mathbb{P}_{X\sim\{\pm1\}^n}(f(X) \neq f(X^{\oplus i})) \\
&= \sum_{i=1}^{n} \mathbb{E}_{X\sim\{\pm1\}^n}[\mathbb{1}_{\{f(X)\neq f(X^{\oplus i})\}}] \\
&= \mathbb{E}_X\left[\sum_{i=1}^{n} \mathbb{1}_{\{f(X)\neq f(X^{\oplus i})\}}\right] \\
&= \mathbb{E}_X[\mathrm{Sens}_f(X)].
\end{aligned}
$$

□

**Remark 1.3.** Recall that

$$\mathrm{Inf}_i(f) = \frac{\#\text{ of sensitive edges in direction } i}{2^{n-1}},$$

so

$$\mathbb{I}(f) = \frac{\#\text{ of sensitive edges}}{2^{n-1}}.$$

Here is a third interpretation of total influence.

**Lemma 1.4.**

$$\mathbb{I}(f) = \sum_{S\subseteq[n]} \widehat{f}(S)^2 \cdot |S|.$$

*Proof.*

$$
\begin{aligned}
\mathbb{I}(f) &= \sum_{i=1}^{n} \mathrm{Inf}_i(f) \\
&= \sum_{i=1}^{n} \sum_{S\ni i} \widehat{f}(S)^2
\end{aligned}
$$

Each coordinate is counted each time it shows up in a set.

$$= \sum_{S\subseteq[n]} \widehat{f}(S)^2 \cdot |S|.$$

□

For any boolean function $f : \{\pm1\}^n \to \{\pm1\}$, we know

$$f(x) = \sum_{S\subseteq[n]} \widehat{f}(S) \cdot \prod_{i\in S} x_i, \qquad \sum_{S\subseteq[n]} \widehat{f}(S)^2 = 1.$$

Define the distribution $\mathscr{S}_f$ over subsets $S \subseteq [n]$, where $S$ is sampled with probability $\widehat{f}(S)^2$.

**Example 1.6.** Recall $\mathrm{MAJ}_3(x_1, x_2, x_3) = \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_1x_2x_3$. This distribution would pick $\{1\}, \{2\}, \{3\}, \{1, 2, 3\}$ each with probability 1/4.

This gives the interpretation

$$\mathbb{I}(f) = \mathbb{E}_{S \sim \mathscr{S}_f}[|S|] = \sum_{S \subseteq [n]} \widehat{f}(S)^2 \cdot |S|.$$

**Remark 1.4.** The distribution $\mathscr{S}_f$ with maximal entropy is the uniform distribution over all subsets. This corresponds to

$$f(x) = x_1x_2 \oplus x_3x_4 \oplus \cdots \oplus x_{n-1}x_n,$$

where we are thinking of the $x_i$ as elements of $\mathbb{F}_2$. Check that $\widehat{f}(S) = \pm 2^{-n/2}$.

By Markov's inequality,

$$\mathbb{P}_{S \sim \mathscr{S}_f}(|S| \geq \underbrace{\mathbb{E}[|S|]}_{\mathbb{I}(f)} \cdot c) \leq \frac{1}{c}.$$

Which boolean function maximizes $\mathbb{I}(f)$? This is the partiy function $\prod_i x_i$. Which boolean function maximizes $\mathrm{Eff}(f)$? It turns out this is the Majority function.

**Proposition 1.2.** *Let $n$ be odd. Among all boolean functions on $n$ variables,* $\mathrm{MAJ}_n$ *is the unique maximizer of* $\mathrm{Eff}(f)$.

*Proof.* Let $f : \{\pm 1\}^n \to \{\pm 1\}$ be any boolean function on $n$ variables.

$$\begin{aligned}
\mathrm{Eff}(f) &= \sum_{i=1}^{n} \widehat{f}(\{i\}) \\
&= \sum_{i=1}^{n} \mathbb{E}_X[f(X)X_i] \\
&= \mathbb{E}_X\left[f(X)\sum_{i=1}^{n} X_i\right].
\end{aligned}$$

To maximize this, we need to take $f(x) = \mathrm{sgn}(\sum_{i=1}^n x_i)$, which is the Majority function. $\square$

Next time, we will given an interpretation of influence in terms of an isoperimetric inequality on the hypercube, and we will prove Arrow's theorem.